Pergamon

Int. J. Solids Structures Vol. 35, No. 14, pp. 1635–1652, 1998

0020-7683/98 $19.00 + .00

PII: S0020-7683(97)00122-4

DUAL ANALYSIS FOR PATH INTEGRALS AND BOUNDS FOR CRACK PARAMETER

CHANG-CHUN WU†, QI-ZHI XIAO
Department of Modern Mechanics, University of Science and Technology of China,
Hefei 230026, China
E-mail: ccwu@sunlx05.mech.ustc.edu.cn

GENKI YAGAWA
Department of Quantum Engineering and Systems Science, The University of Tokyo, 7-3-1,
Hongo, Bunkyo-ku, Tokyo 113, Japan
E-mail: yagawa@garlic.gen.u-tokyo.ac.jp

(*Received* 3 *September* 1996; *in revised form* 15 *April* 1997)

Abstract—A dual analysis for the path integrals is carried out theoretically and numerically. As a dual form of Rice's J-integral, an alternative path-independent I^*-integral is suggested in the paper, which can be formulated as the complementary energy release rate for the linear/nonlinear elasticity fracture system with a blunting crack model. As an energy functional of the stress and displacement, I^* is equivalent to J in value since the effect of the stress distributions on the front of crack is included in its formulation. Dealing with bound estimation problems for crack parameters, the upper and lower bound theorems are described, respectively. I^* is useful by the fact that its approximate solution is able to provide an upper bound for the exact one, and that it will enable the hybrid finite element to be a power role in fracture calculations. A series of numerical results is offered to verify the points mentioned in the paper. © 1998 Elsevier Science Ltd.

1. INTRODUCTION

The well-known J-integral (Rice, 1968; Eshelby, 1956) has been proved to be the most valuable fracture parameter for the linear/nonlinear elastic fracture mechanics. As the generalized manners of J, a series of path-independent integrals related to the potential energy of crack systems have been presented by Knowles and Sternberg (1972) and Atluri (1982), many new path independent integrals were developed for linear elasticity by Tsamasphyros and Theocaris (1982) and the proof of existence of an infinite number path independent integrals was given by Olver (1984). Some reviews have offered by Kanninen and Popelar (1985) and Hellen and Blackburn (1986). In the mechanical point of view, however, there must exist a dual form of J-integral which should be a path-independent integral related to the complementary energy of the crack system. Based on the above recognition, Bui (1974) presented an I-integral, which is expressible as the rate of decrease of the energy functional for the stress field. In previous decades, the I-integral was, however, neither widely applied nor deeply investigated either. The idea for dual path independent integrals offered by Bui should be rational and considerable, but some problems related to I should be cleared up and developed further. As shown in this paper, the value of I is not identical to that of J, and the mechanical sense of I is different from that of J as well.

In this paper, an alternating path-independent I^*-integral is presented, which can be considered as the complementary energy release rate of a fracture system with a blunting crack model. The duality of J and I^* can be verified theoretically and numerically. A fascinating application of I^* would be that it is able to provide an approximate upper bound for the exact I^*-integral (to be identical with the exact J-integral). With respect to the calculations of the path integrals, it can be demonstrated (Wu and Pian, 1997) that as the energy functional of displacements, $J = J(u_i)$ should be calculated by the assumed displacement finite elements, which always provide some locking problems in the fully plastic analysis (Nagtegaal *et al.*, 1974). On the other hand, as the functional of stresses

† Corresponding author.

and displacements, $I^* = I^*(\sigma_{ij}, u_i)$ can be calculated by the hybrid finite elements with the independent stresses σ_{ij} and the displacements u_i, thus a numerical solution without locking problem for I^* is available even in the case of the fully plastic fracture analysis.

2. DEFINITION AND ATTRIBUTE

Let us consider a homogeneous cracked system of lienar or nonlinear elsatic material as shown in Fig. 1, in which $o\ x_1x_2$ are the Cartesian coordinates and the curve Γ is selected as an integral path, $\bar{T}_i$ is the prescribed traction force on the boundary S_σ, the prescribed displacement $\bar{u}_i = 0$ on the boundary S_u, and n_j is the direction cosine for the normal of system edge. Then the present path integral can be defined as

$$I^* = \int_\Gamma \left[-B(\sigma_{ij})\,\mathrm{d}x_2 + u_i \frac{\partial \sigma_{ij}}{\partial x_1} n_j\,\mathrm{d}s + \frac{\partial}{\partial x_j}(u_i\sigma_{i2})\,\mathrm{d}x_j \right] \tag{1}$$

where $B(\sigma_{ij})$ is the complementary energy function. $I^* = I^*(\sigma_{ij}, u_i)$ is an energy functional with two kinds of field variables: the stress σ_{ij} and the displacement u_i.

The definition of I^* in eqn (1) is based on the following assumptions:

The strain–stress law is given by

$$\varepsilon_{ij} = \frac{\partial B}{\partial \sigma_{ij}}. \tag{2}$$

The strain–displacement relation should be

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{3}$$

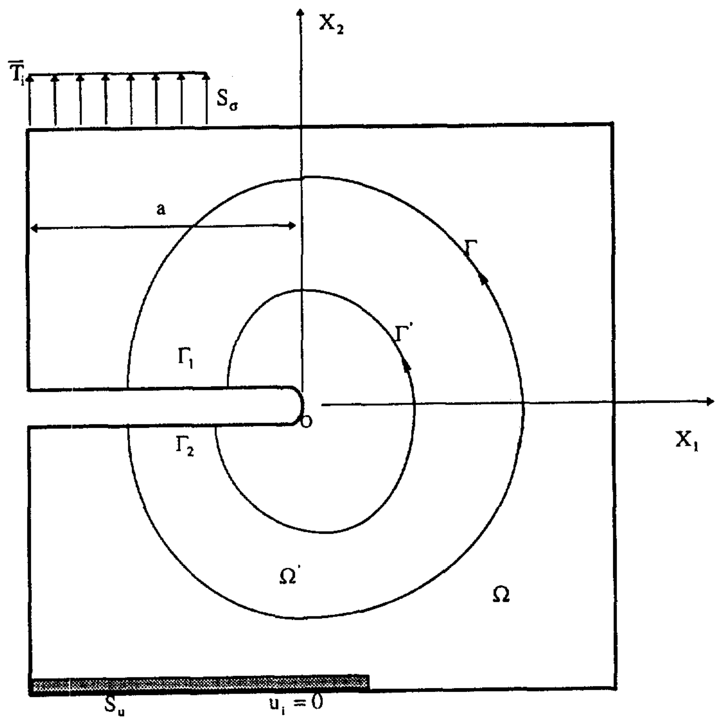

Fig. 1. Alternative integral contours for two-dimensional crack system.

The stress field should be in equilibrium, i.e.

$$\sigma_{ij,j} = 0 \tag{4a}$$

in Ω (free of body forces) and

$$\sigma_{ij} n_j = T_i = 0 \tag{4b}$$

on the crack surface (free of tractions).

It can be demonstrated that the present I^*-integral given by eqn (1) is path independent and is equivalent to Rice's J-integral. In the first place, we prove I^* is a path independent integral as follows.

Taking two deferent curves Γ and Γ' as indicated in Fig. 1, with the conditions (2)–(4) and Green's formula, we have

$$\oint -B(\sigma_{ij})\,\mathrm{d}x_2 = \int_{\Omega'} -\frac{\partial}{\partial x_1}B(\sigma_{ij})\,\mathrm{d}\Omega = -\int_{\Omega'} \varepsilon_{ij}\frac{\partial \sigma_{ij}}{\partial x_1}\,\mathrm{d}\Omega \tag{5}$$

$$\oint u_i\left(\frac{\partial \sigma_{ij}}{\partial x_1}\right) n_j\,\mathrm{d}s = \int_{\Omega'} \frac{\partial}{\partial x_j}\left(u_i\frac{\partial \sigma_{ij}}{\partial x_1}\right)\mathrm{d}\Omega = \int_{\Omega'} \frac{\partial u_i}{\partial x_j}\frac{\partial \sigma_{ij}}{\partial x_1}\,\mathrm{d}\Omega \tag{6}$$

$$\begin{aligned}\oint \frac{\partial}{\partial x_j}(u_i\sigma_{i2})\,\mathrm{d}x_j &= \oint\left[-\frac{\partial}{\partial x_1}(u_i\sigma_{i2})n_2 + \frac{\partial}{\partial x_2}(u_i\sigma_{i2})n_1\right]\mathrm{d}s \\ &= \int_{\Omega'}\left[-\frac{\partial^2}{\partial x_1\partial x_2}(u_i\sigma_{i2}) + \frac{\partial^2}{\partial x_2\partial x_1}(u_i\sigma_{i2})\right]\mathrm{d}\Omega \\ &= 0\end{aligned} \tag{7}$$

such that

$$\oint\left[-B(\sigma_{ij})\,\mathrm{d}x_2 + u_i\left(\frac{\partial \sigma_{ij}}{\partial x_1}\right)n_j\,\mathrm{d}s + \frac{\partial}{\partial x_j}(u_i\sigma_{i2})\,\mathrm{d}x_j\right] = 0. \tag{8a}$$

The above identify can be expressed as

$$\oint(\ldots) = \int_{\partial\Omega'}(\ldots) = \int_{\Gamma+\Gamma_1-\Gamma'+\Gamma_2}(\ldots) = 0. \tag{8b}$$

Note that on $\Gamma_1+\Gamma_2$: $\mathrm{d}x_2 = 0$ and $\sigma_{ij}n_j = 0$. It turns out that $\sigma_{i2}n_2 = 0$.

Finally we have

$$\int_{\Gamma}(\ldots) - \int_{\Gamma'}(\ldots) = 0, \quad \text{i.e. } I^*_{\text{path}\Gamma} = I^*_{\text{path}\Gamma'}. \tag{9}$$

In the second place, let us prove $I^* = J$, where

$$J(u_i) = \int_{\Gamma}\left[W(\varepsilon_{ij})\,\mathrm{d}x_2 - \sigma_{ij}n_j\left(\frac{\partial u_i}{\partial x_1}\right)\mathrm{d}s\right] \tag{10}$$

is Rice's J-integral, wherein $W(\varepsilon_{ij})$ is the strain energy function.

Referring to the path Γ in Fig. 1,

$$
\begin{aligned}
J - I^* &= \int_\Gamma [W(\varepsilon_{ij}) + B(\sigma_{ij})]\,dx_2 \\
&\quad - \int_\Gamma \left[\sigma_{ij}n_j\left(\frac{\partial u_i}{\partial x_1}\right) ds + u_i\left(\frac{\partial \sigma_{ij}}{\partial x_1}\right) n_j\,ds + \frac{\partial}{\partial x_j}(u_i\sigma_{i2})\,dx_j\right] \\
&= \int_\Gamma \frac{\partial}{\partial x_j}(u_i\sigma_{ij})\,dx_2 - \int_\Gamma \frac{\partial}{\partial x_1}(u_i\sigma_{i1})\,dx_2 + \int_\Gamma \frac{\partial}{\partial x_1}(u_i\sigma_{i2})\,dx_1 - \int_\Gamma \frac{\partial}{\partial x_j}(u_i\sigma_{i2})\,dx_j \\
&= \int_\Gamma \frac{\partial}{\partial x_2}(u_i\sigma_{i2})\,dx_2 + \int_\Gamma \frac{\partial}{\partial x_1}(u_i\sigma_{i2})\,dx_1 - \int_\Gamma \frac{\partial}{\partial x_j}(u_i\sigma_{i2})\,dx_j \\
&= 0. \qquad (11)
\end{aligned}
$$

This means that the exact I^* is identical to the exact J in value. Therefore, I^*-integral can be used as a control parameter for the linear/nonlinear elastic fracture problem as J-integral can.

3. COMPLEMENTARY ENERGY RELEASE RATE

It can be verified that the present path independent integral I^* can be expressed as the complementary energy release rate for a given crack system, and we have

$$I^* = \frac{d\Pi_c}{da} \qquad (12a)$$

where a is the crack length, and the system complementary energy is given by

$$\Pi_c(\sigma_{ij}) = \int_\Omega B(\sigma_{ij})\,d\Omega - \int_{S_u} \bar{u}_i\sigma_{ij}n_j\,ds \qquad (12b)$$

where $\bar{u}_i$ is the prescribed displacement on the boundary S_u.

Observing the system with a blunting model shown in Fig. 2, the external bound of the crack system can be selected as an integral contour due to the path independent property

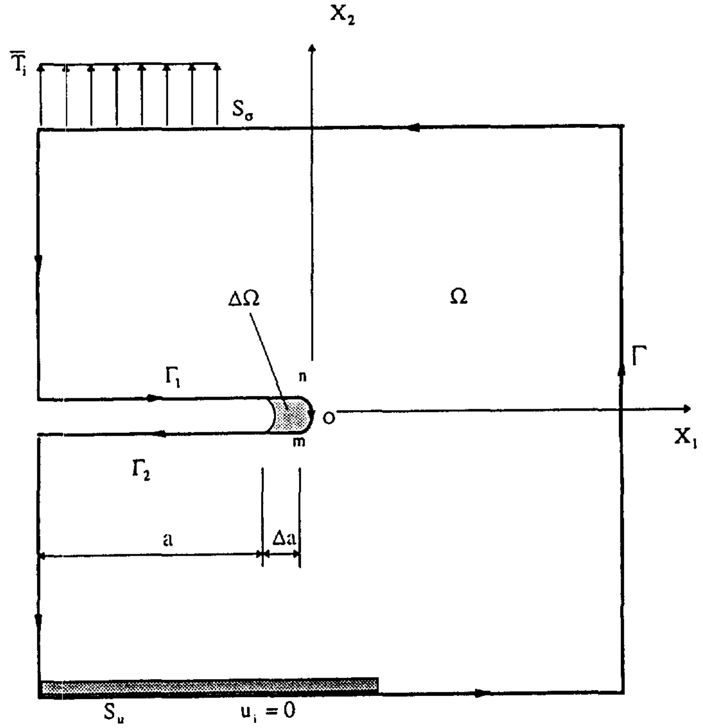

Fig. 2. Blunting crack model with different crack lengths.

of I^*. In this case, $\partial\Omega = \Gamma+\Gamma_1+\Gamma_2+nm$ and $nm = -mn$ indicates the front of the crack, i.e., the tip of a smooth ended notch, as Rice (1968) termed. Assume that the crack length a has grown by an amount Δa, then the coordinates located at the crack top are also moved by Δa in the x_1-direction. Therefore, both effects related with the variations of "a" and "x_1" should be included in the calculation of $\mathrm{d}\Pi_c/\mathrm{d}a$. Besides, the change of effective area of the cracked system, i.e. the shaded one in Fig. 2, will produce a decrease of the complementary energy by an amount of

$$\int_{\Delta\Omega} B(\sigma_{ij})\,\mathrm{d}\Omega \overset{(\Delta a\to 0)}{=} \Delta a\int_m^n B(\sigma_{ij})\,\mathrm{d}x_2. \tag{13}$$

Thus, corresponding to the crack extention by Δa, the increment of the system complementary energy should be of the form

$$\Delta\Pi_c(a,x_1) = \frac{\partial\Pi_c}{\partial a}\Delta a+\frac{\partial\Pi_c}{\partial x_1}\Delta x_1-\Delta a\int_m^n B(\sigma_{ij})\,\mathrm{d}x_2 \tag{14}$$

and

$$\begin{aligned}\frac{\mathrm{d}\Pi_c}{\mathrm{d}a} &= \lim_{\Delta a\to 0}\frac{\Delta\Pi_c}{\Delta a} = \frac{\partial\Pi_c}{\partial a}+\frac{\partial\Pi_c}{\partial x_1}\frac{\partial x_1}{\partial a}-\int_m^n B(\sigma_{ij})\,\mathrm{d}x_2\\ &= \frac{\partial\Pi_c}{\partial a}-\frac{\partial\Pi_c}{\partial x_1}-\int_m^n B(\sigma_{ij})\,\mathrm{d}x_2.\end{aligned} \tag{15}$$

In the following development, it is convenient to use the complementary virtual work principle, which states that for the stress field satisfying the equilibrium equation and the prescribed traction boundary condition, there exsits the energy identity

$$\int_{S_u}\bar{u}_i\sigma_{ij}n_j\,\mathrm{d}s = \int_\Omega \varepsilon_{ij}\sigma_{ij}\,\mathrm{d}\Omega \quad \text{where} \quad \varepsilon_{ij} = \frac{\partial B}{\partial\sigma_{ij}}. \tag{16}$$

Since $\partial\sigma_{ij}/\partial a$ and $\partial\sigma_{ij}/\partial x_1$ are also in equilibrium, eqn (16) can be, respectively, developed into the following forms:

$$\int_{S_u}\bar{u}_i\frac{\partial\sigma_{ij}}{\partial a}n_j\,\mathrm{d}s = \int_\Omega \varepsilon_{ij}\frac{\partial\sigma_{ij}}{\partial a}\mathrm{d}\Omega \quad\text{and}\quad \int_{S_u}\bar{u}_i\frac{\partial\sigma_{ij}}{\partial x_1}n_j\,\mathrm{d}s = \int_\Omega \varepsilon_{ij}\frac{\partial\sigma_{ij}}{\partial x_1}\mathrm{d}\Omega. \tag{17}$$

In views of eqns (12b) and (17), we have

$$\begin{aligned}\frac{\partial\Pi_c}{\partial a} &= \int_\Omega\frac{\partial B}{\partial\sigma_{ij}}\frac{\partial\sigma_{ij}}{\partial a}\mathrm{d}\Omega-\int_{S_u}\bar{u}_i\frac{\partial\sigma_{ij}}{\partial a}\mathrm{n}_j\,\mathrm{d}s\\ &= \int_\Omega\varepsilon_{ij}\frac{\partial\sigma_{ij}}{\partial a}\mathrm{d}\Omega-\int_{S_u}\bar{u}_i\frac{\partial\sigma_{ij}}{\partial a}n_j\,\mathrm{d}s = 0\end{aligned} \tag{18}$$

and we have also

$$\begin{aligned}\frac{\partial\Pi_c}{\partial x_1} &= \int_\Omega\frac{\partial B}{\partial x_1}\mathrm{d}\Omega-\int_{S_u}\bar{u}_i\frac{\partial\sigma_{ij}}{\partial x_1}n_j\,\mathrm{d}s\\ &= \oint B(\sigma_{ij})n_1\,\mathrm{d}s-\int_\Omega\varepsilon_{ij}\frac{\partial\sigma_{ij}}{\partial x_1}\mathrm{d}\Omega\end{aligned}$$

$$= \oint B\,\mathrm{d}x_2 - \oint u_i \frac{\partial \sigma_{ij}}{\partial x_1} n_j\,\mathrm{d}s$$

$$= \int_\Gamma B\,\mathrm{d}x_2 - \int_m^n B\,\mathrm{d}x_2 - \int_\Gamma u_i \frac{\partial \sigma_{ij}}{\partial x_1} n_j\,\mathrm{d}s + \int_m^n u_i \frac{\partial \sigma_{ij}}{\partial x_1} n_j\,\mathrm{d}s. \tag{19}$$

Observing that the front of the crack keeps a stationary configuration regardless of the changes of crack length in the assumed crack extension, the following geometrical constraint holds:

$$\partial u_1/\partial x_2 = 0 \quad \text{on} \quad mn. \tag{20}$$

Under the condition (20), a stress analysis on the crack front mn was carried out (Xiao, 1996), and the following identity can always be obtained regardless of the stress distributions near the crack tip are symmetric or anti-symmetric:

$$\int_m^n u_i \frac{\partial \sigma_{ij}}{\partial x_1} n_j\,\mathrm{d}s = \int_m^n \frac{\partial}{\partial x_j}(u_i \sigma_{i2})\,\mathrm{d}x_j. \tag{21}$$

Additionally, by using the identity (7), we have

$$\oint \frac{\partial}{\partial x_j}(u_i \sigma_{i2})\,\mathrm{d}x_j = \int_\Gamma \frac{\partial}{\partial x_j}(u_i \sigma_{i2})\,\mathrm{d}x_j - \int_m^n \frac{\partial}{\partial x_j}(u_i \sigma_{i2})\,\mathrm{d}x_j = 0. \tag{22}$$

Introducing eqns (21) and (22) into eqn (19),

$$\frac{\partial \Pi_c}{\partial x_1} = \int_\Gamma B\,\mathrm{d}x_2 - \int_m^n B\,\mathrm{d}x_2 - \int_\Gamma u_i \frac{\partial \sigma_{ij}}{\partial x_1} n_j\,\mathrm{d}s - \int_\Gamma \frac{\partial}{\partial x_j}(u_i \sigma_{i2})\,\mathrm{d}x_j. \tag{23}$$

Substitution of eqns (18) and (23) into eqn (15) finally produces

$$\frac{\mathrm{d}\Pi_c}{\mathrm{d}a} = -\int_\Gamma B\,\mathrm{d}x_2 + \int_\Gamma u_i \frac{\partial \sigma_{ij}}{\partial x_1} n_j\,\mathrm{d}s + \int_\Gamma \frac{\partial}{\partial x_j}(u_i \sigma_{i2})\,\mathrm{d}x_j = I^*. \tag{24}$$

4. DISCUSSION ON *I*-INTEGRAL

The *I*-integral suggested by Bui (1974) takes the form of

$$I = \int_\Gamma \left[-B(\sigma_{ij})\,\mathrm{d}x_2 + u_i \frac{\partial \sigma_{ij}}{\partial x_1} n_j\,\mathrm{d}s \right]. \tag{25}$$

In comparison with the present I^*-integral defined by eqn (1), the following term is lost in the *I*-integral:

$$\Delta I = \int_\Gamma \frac{\partial}{\partial x_j}(u_i \sigma_{i2})\,\mathrm{d}x_j. \tag{26}$$

It should be cleared up as to what is the sense of the above term. Observing that

$$\frac{\partial}{\partial x_j}(u_i\sigma_{i2})\,\mathrm{d}x_j = -\frac{\partial}{\partial x_1}(u_i\sigma_{i2})n_2\,\mathrm{d}x + \frac{\partial}{\partial x_2}(u_i\sigma_{i2})\,\mathrm{d}x_2 \tag{27}$$

$$\sigma_{ij}\varepsilon_{ij}n_1\,\mathrm{d}s = \sigma_{ij}u_{i,j}n_1\,\mathrm{d}s = \frac{\partial}{\partial x_1}(u_i\sigma_{i1})n_1\,\mathrm{d}s + \frac{\partial}{\partial x_2}(u_i\sigma_{i2})\,\mathrm{d}x_2. \tag{28}$$

Then, by using the traction free condition (4b) to the front of crack mn, we have

$$(28)-(27) = \frac{\partial}{\partial x_1}(u_i\delta_{i1}n_1 + u_i\sigma_{i2}n_2)\,\mathrm{d}s = \frac{\partial}{\partial x_1}(u_i\sigma_{ij}n_j)\,\mathrm{d}s = 0.$$

Therefore,

$$\int_m^n \frac{\partial}{\partial x_j}(u_i\sigma_{i2})\,\mathrm{d}x_j = \int_m^n \sigma_{ij}\varepsilon_{ij}n_1\,\mathrm{d}x = 2\int_m^n B(\sigma_{ij})\,\mathrm{d}x. \tag{29}$$

On the other hand, by using the identity (7) and $\mathrm{d}x_2 = 0$ on $\Gamma_1+\Gamma_2$,

$$\int_\Gamma \frac{\partial}{\partial x_j}(u_i\sigma_{i2})\,\mathrm{d}x_j = \int_m^n \frac{\partial}{\partial x_j}(u_i\sigma_{i2})\,\mathrm{d}x_j. \tag{30}$$

Substitution of eqns (29) and (30) into eqn (26) results in

$$\Delta I = \int_\Gamma \frac{\partial}{\partial x_j}(u_i\sigma_{i2})\,\mathrm{d}x_j = 2\int_m^n B(\sigma_{ij})\,\mathrm{d}x_2. \tag{31}$$

It is clear that the term ΔI is just the double of complementary energy distributed on the front of the crack.

With regard to the blunting crack model as shown in Fig. 2, undoubtedly, $\Delta I \neq 0$, and it cannot be ignored. Further, let us inspect a sharp crack model, which can be considered as a limit form of the blunted crack. When the front of the crack is shrunk to a point, i.e. the crack tip, the integral path $mn \to 0$. Simultaneously, however, the stresses at the crack tip become infinite ($\sigma_{ij} \to \infty$), and so will the complementary energy function, i.e. $B(\sigma_{ij}) \to \infty$ with the $1/r$ singularity for the subject of nonlinear elasticity. In this situation, instead of eqn (31) we have

$$\Delta I = \lim_{mn\to 0} 2\int_m^n B(\sigma_{ij})\,\mathrm{d}x_2 = 0\times\infty. \tag{32}$$

It is a non-zero indefinite value and cannot be ignored either. In general, the term ΔI provides the path integral with a considerable energy contributions due to the non-zero stress distributions on the front of the crack, and it can never be ignored no matter which kind of crack models is to be considered.

The equivalence of I^* and J shown in eqn (11) clearly shows $I \neq J$ for an exact solution satisfying the eqns (2)–(4). Thus, the integrals I and J possess some different mechanical meanings. For instance, in the linear elasticity or small-scale yielding case, the integral J or I^* is related to the stress intensity factor K_I by

$$I^* = J = K_\mathrm{I}^2/E' \tag{33}$$

where $E' = E$ (Young's modulus) for the plane stress status, or $E' = E/(1-\mu^2)$ for the plane strain one. Between the I-integral and K_I, however, maybe it is hard to find a relationship to be like eqn (33).

5. UPPER/LOWER BOUND PROBLEM

In view of the complexity of nonlinear fracture mechanics, it is hard to get exact solutions for both of J and I^*, no matter what experimental methods or numerical methods are used. Therefore, a technique to determine the upper and lower bounds of the path independent integrals becomes of much concern here (Bui, 1974; Atluri *et al.*, 1984). He and Hutchinson (1983) presented some results about the bounds of J for fully plastic crack problems. However it is hard to apply them to general/practical situations as only the infinite bodies are considered. In the next, two bound theories for J and I^* will, respectively, be established in a dual form. The discussion will be limited in the linear/nonlinear elasticity with small strain assumption.

Lower bound theorem

For a given elastic cracked system with the boundary constraint $\bar{u}_i|_{S_u} = 0$, if u_i and $\tilde{u}_i$ are, respectively, the exact displacement and the approximate one based on the minimum potential energy principle, the approximate value of J-integral will take the lower bound of its exact one:

$$J(\tilde{u}_i) \leqslant J(u_i). \tag{34}$$

Proof. Let $\tilde{u}_i = u_i + \delta u_i$, δu_i are compatible virtual displacements. Then the approximate potential energy can be expressed as (Washizu, 1975)

$$\Pi_P(\tilde{u}_i) = \Pi_P(u_i) + \delta\Pi_P + \delta^2\Pi_P. \tag{35}$$

As a stationary condition of $\Pi_P(\tilde{u}_i)$, $\delta\Pi_p = 0$. On the other hand, for the given exact displacements u_i which satisfied the boundary condition $\bar{u}_i|_{S_u} = 0$,

$$\int_{S_\sigma} u_i \bar{T}_i \, \mathrm{d}s = \int_\Omega \varepsilon_{ij}\sigma_{ij} \, \mathrm{d}\Omega$$

should be holding, such that

$$\Pi_P(u_i) = \int_\Omega W(u_i) \, \mathrm{d}\Omega - \int_{S_\sigma} u_i \bar{T}_i \, \mathrm{d}s = -\int_\Omega W(u_i) \, \mathrm{d}\Omega. \tag{36}$$

Besides it is well known that

$$\delta^2\Pi_P = \int_\Omega W(\delta u_i) \, \mathrm{d}\Omega \geqslant 0. \tag{37}$$

In accordance with the definition of J-integral and its positive definite attribute, we have

$$J(\tilde{u}_i) = -\frac{\mathrm{d}}{\mathrm{d}a}\Pi_P(\tilde{u}_i) = J(u_i) + \delta^2 J \tag{38}$$

where

$$J(u_i) = -\frac{\mathrm{d}}{\mathrm{d}a}\Pi_P(u_i) = \frac{\mathrm{d}}{\mathrm{d}a}\int_\Omega W(u_i) \, \mathrm{d}\Omega \geqslant 0 \tag{39}$$

$$\delta^2 J = -\frac{\mathrm{d}}{\mathrm{d}a}\delta^2\Pi_P = -\frac{\mathrm{d}}{\mathrm{d}a}\int_\Omega W(\delta u_i) \, \mathrm{d}\Omega. \tag{40}$$

Observing that $W(u_i)$ and $W(\delta u_i)$ are all the deformation energy functions, they possess the same function configurations, and must take the same varying regularity when a stable crack in the system is developing. Thus, we see by comparison of eqns (39) and (40) that $\delta^2 J \leqslant 0$, and then the inequality (34) must be true.

Upper bound theorem

For a given elastic cracked system with the boundary constraint $\bar{u}_i|_{S_u} = 0$, if σ_{ij} and $\tilde{\sigma}_{ij}$ are, respectively, the exact stress and the approximate one based on the minimum complementary energy principle, the approximate value of I^*-integral will take the upper bound of its exact one:

$$I^*(\tilde{\sigma}_{ij}) \geqslant I^*(\sigma_{ij}). \tag{41}$$

Proof. Let $\tilde{\sigma}_{ij} = \sigma_{ij} + \delta\sigma_{ij}$, where $\delta\sigma_{ij}$ are virtual stresses

$$\Pi_c(\tilde{\sigma}_{ij}) = \Pi_c(\sigma_{ij}) + \delta\Pi_c + \delta^2\Pi_c. \tag{42}$$

As a stationary condition of $\Pi_c(\tilde{\sigma}_{ij})$, we have $\delta\Pi_c = 0$. On the other hand, due to $\bar{u}_i|_{S_u} = 0$, the complimentary energy functional (12b) is now

$$\Pi_c(\sigma_{ij}) = \int_\Omega B(\sigma_{ij})\,d\Omega. \tag{43}$$

Besides,

$$\delta^2\Pi_c = \int_\Omega B(\delta\sigma_{ij})\,d\Omega \geqslant 0. \tag{44}$$

In respect to the exact solutions, $I^*(\sigma_{ij}) = J(u_i) \geqslant 0$. Thus, by means of eqns (12a) and (42)–(44), we have

$$I^*(\tilde{\sigma}_{ij}) = I^*(\sigma_{ij}) + \delta^2 I^* \tag{45}$$

$$I^*(\sigma_{ij}) = \frac{d}{da}\Pi_c(\sigma_{ij}) = \frac{d}{da}\int_\Omega B(\sigma_{ij})\,d\Omega \geqslant 0 \tag{46}$$

$$\delta^2 I^* = \frac{d}{da}\delta^2\Pi_c = \frac{d}{da}\int_\Omega B(\delta\sigma_{ij})\,d\Omega. \tag{47}$$

Observing that both $B(\sigma_{ij})$ and $B(\delta\sigma_{ij})$ are the positive definite complementary energy functions, and they possess the same function configurations. Therefore, they must take the same varying regularity when the system is suffering from a stable crack development. By comparison of eqns (46) and (47), we find that $\delta^2 I^* \geqslant 0$, and then the inequality eqn (41) must be true.

6. NUMERICAL DEMONSTRATION

In order to demonstrate the path independent property of I^*, some typical crack specimens are considered, they include

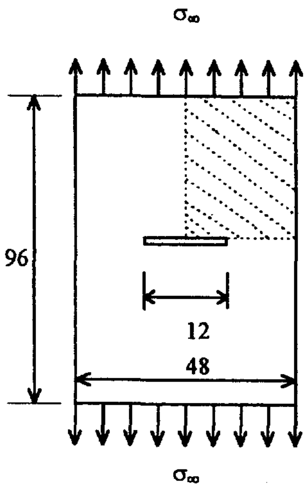

Fig. 3. Center-cracked panel (CCP) with uniform stretching load σ_∞.

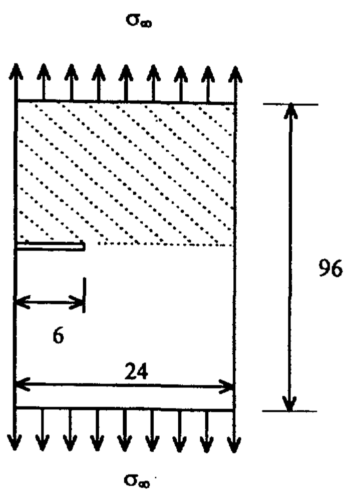

Fig. 4. Single-edge cracked panel (SECP) with uniform stretching load σ_∞.

- CCP is the center cracked panel with uniform stretching load σ_∞ (Fig. 3)
- SECP is the single edge cracked panel with uniform stretching load σ_∞ (Fig. 4)
- DECP is the double edge cracked panel with unifrom stretching load σ_∞ (Fig. 5) and
- CBB is the cracked beam bending with concentrated load P (Fig. 6).

The adopted discrete finite element meshes (only for the shaded parts of each specimen) and the selected integral contours are shown in Figs 7–9 respectively. Material constants of Young's modulus E and Poisson's ratio μ are 1.0 and 0.49, respectively. Two cases of plane stress and plane strain are considered for every specimens which are calculated by using I^*-integral (1) and I-integral (25) simultaneously.

Since both I^* and I are the energy functional with two field (σ_{ij}, u_i), it is rational to employ two-field hybrid finite element approaches in our numerical calculations. We select the 4-node plane hybrid model termed as P–S element (Pian and Sumihara, 1984) due to its excellent behavior. In our linear elastic crack problems, $\sigma_\infty = 1.0$, all the obtained solutions of I^* and I are transferred to the stress intensity factor K_I by means of eqn (33). Figures 10–13 provide a series of numerical results for the individual specimens wherein

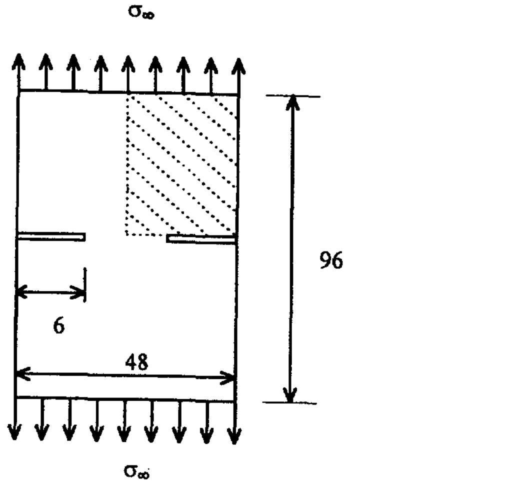

Fig. 5. Double-edge cracked panel (DECP) with uniform stretching load σ_∞.

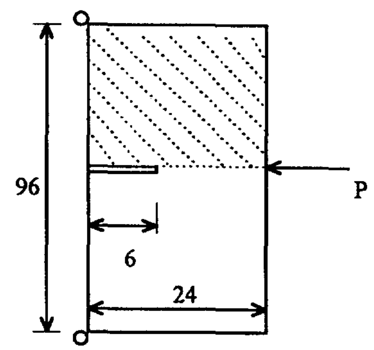

Fig. 6. Three-point cracked beam bending (CBB) with concentrated load P.

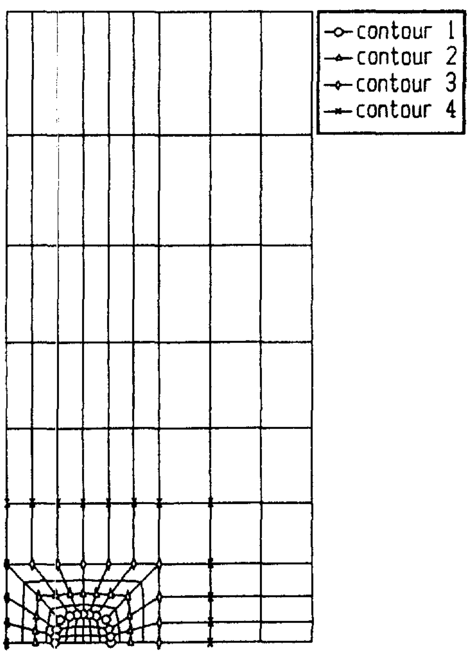

Fig. 7. Finite element meshes for CCP/SECP and the selected contours.

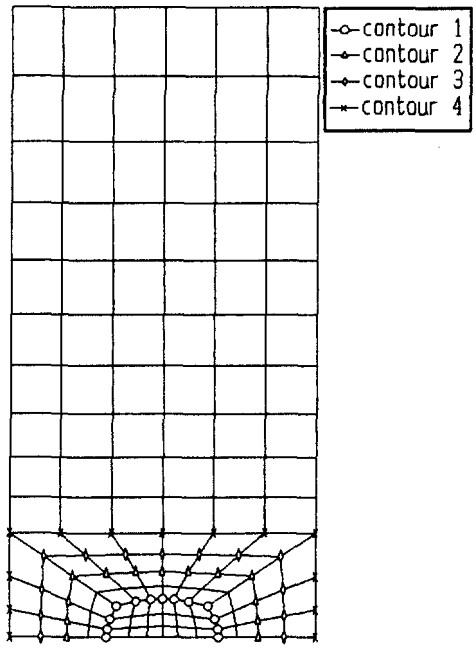

Fig. 8. Finite element meshes for DECP and the selected contours.

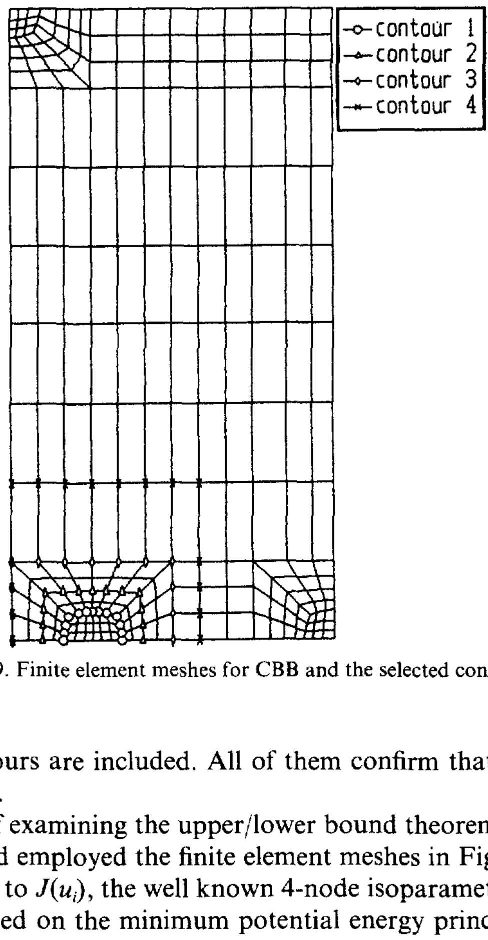

Fig. 9. Finite element meshes for CBB and the selected contours.

four independent contours are included. All of them confirm that the present I^*-integral (1) is path independent.

For the purpose of examining the upper/lower bound theorems, we took CCP in Fig. 3 as a test specimen and employed the finite element meshes in Fig. 14. In order to obtain a lower bound solution to $J(u_i)$, the well known 4-node isoparametric element, termed Q4, was used since it is based on the minimum potential energy principle in accordance with the lower bound theorem (34). On the other hand, in order to obtain a upper bound solution to $I^*(\sigma_{ij})$, we need an equilibrium element which satisfied the equilibrium equations and is in accordance with the complementary energy principle and the upper bound theorem (41). To this end, we imposed the equilibrium equations to the P–S hybrid element by the

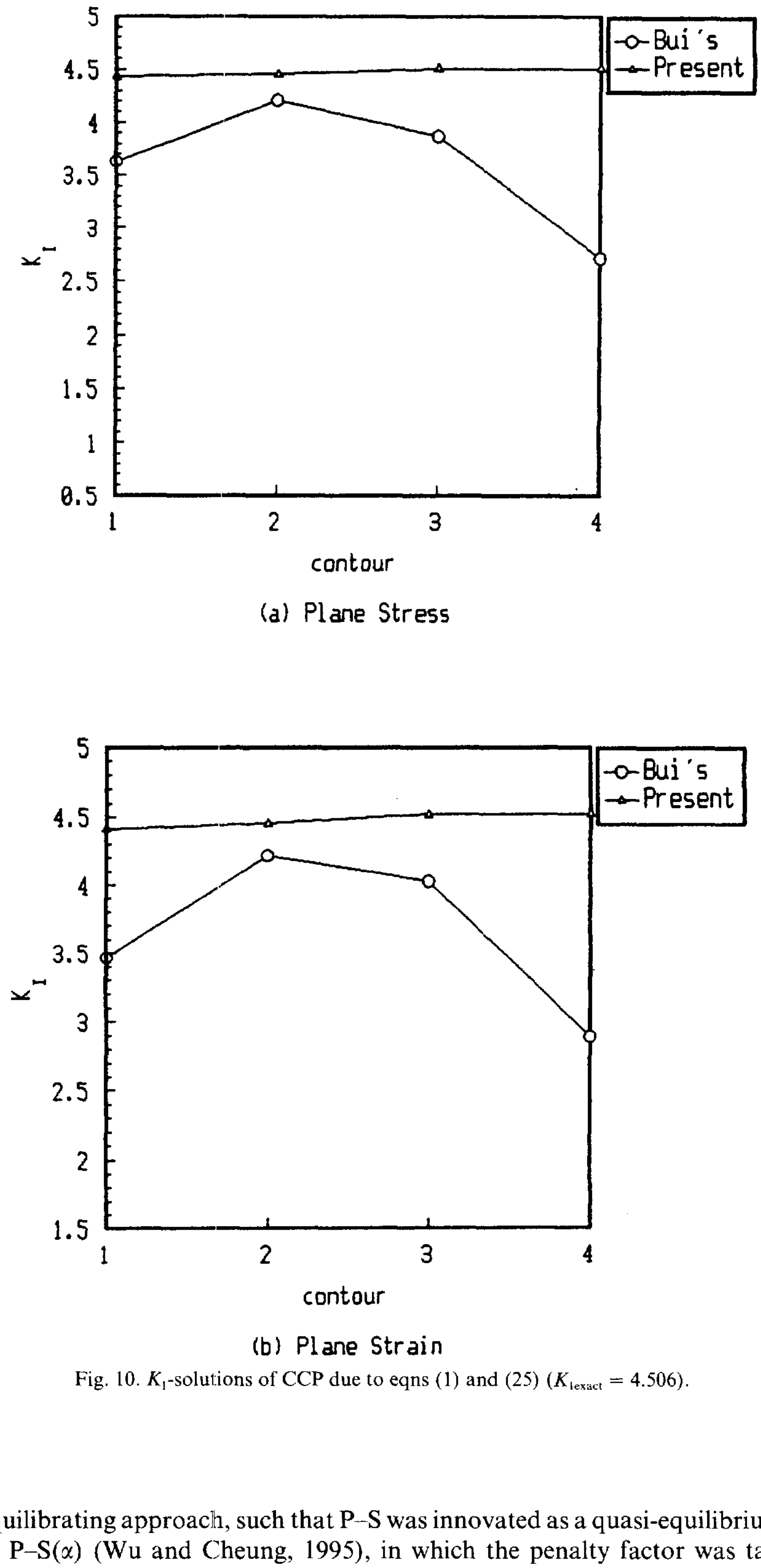

Fig. 10. K_I-solutions of CCP due to eqns (1) and (25) ($K_{Iexact} = 4.506$).

penalty equilibrating approach, such that P–S was innovated as a quasi-equilibrium element termed as P–S(α) (Wu and Cheung, 1995), in which the penalty factor was taken to be $\alpha = 10^4$. Tables 1 and 2 exhibit some approximate solutions of the stress intensity factor K_I corresponding to J and I^*, and the exact value of K_I was given by Ewalds and Wanhill (1984). We see that the formulas (1) and (10), respectively, produce the upper and lower bounds to the exact integral values, in both the plane stress and the plane strain cases, and regardless of the selection of integral paths.

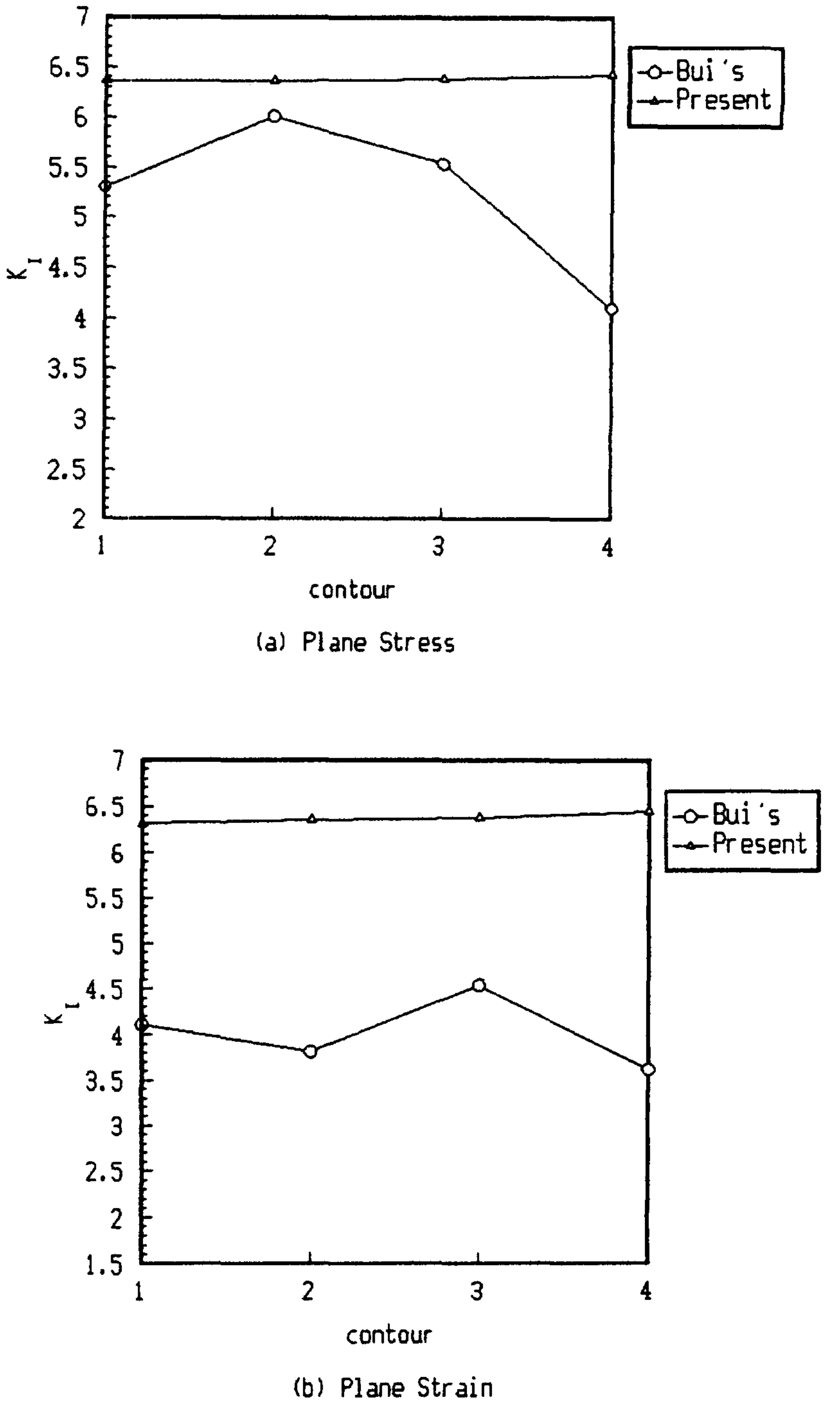

Fig. 11. K_I-solutions of SECP due to eqns (1) and (25) ($K_{Iexact} = 6.517$).

Table 1. Stress intensity factor (K_I) solutions of CCP specimen (Fig. 3) in plane stress case

Integral path (Fig. 14)	1	2	3	4
Lower bound solutions by $J(10)$	4.408	4.485	4.477	4.475
Upper bound solutions by $I^*(1)$	4.525	4.562	4.553	4.553
Exact K_I	4.506	4.506	4.506	4.506

($E = 1.0$, $\mu = 0.3$, $\sigma_\infty = 1.0$).

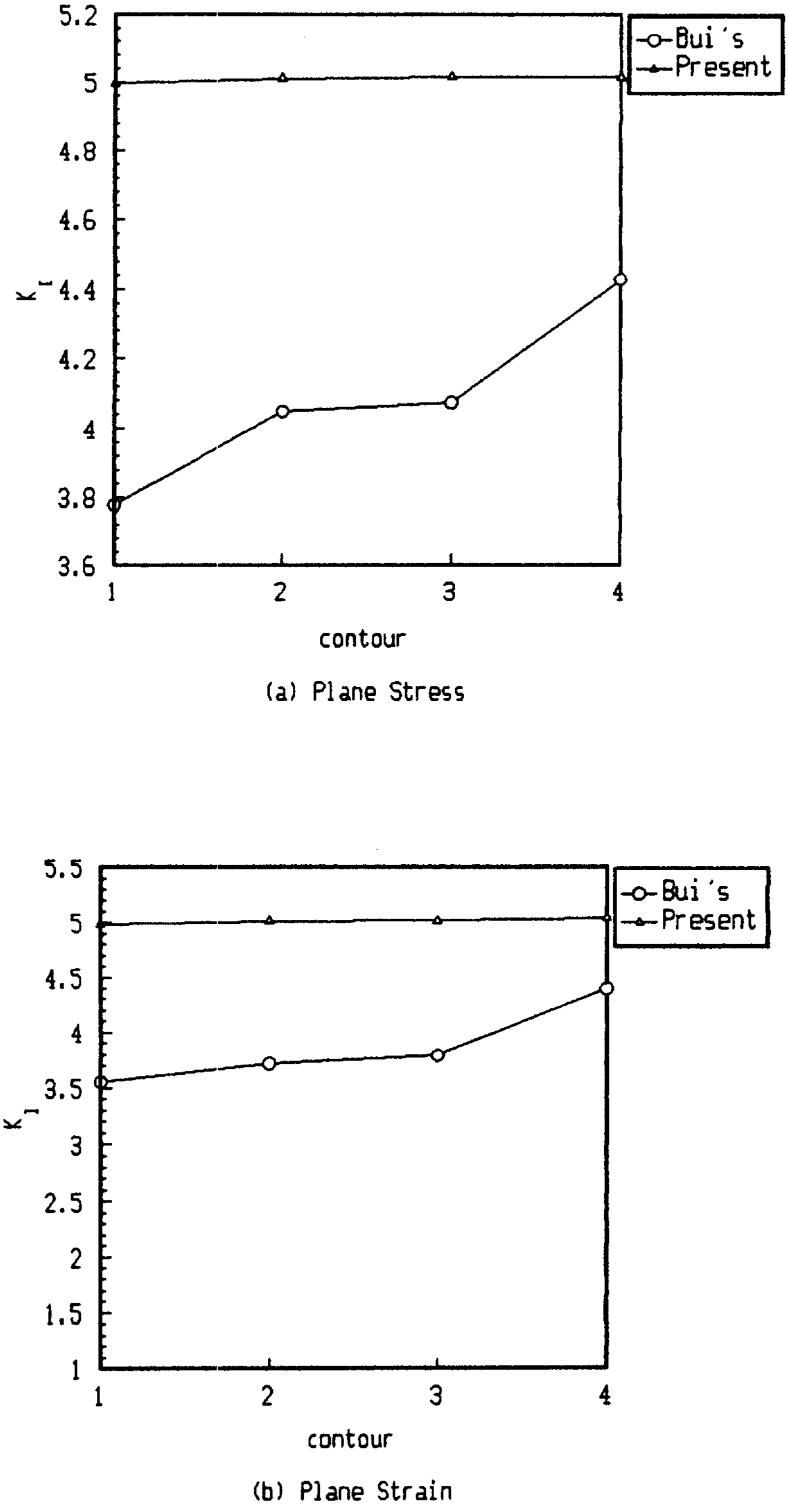

Fig. 12. K_I-solutions of DECP due to eqns (1) and (25) (K_{Iexact} = 4.900).

Table 2. Stress intensity factor (K_I) solutions of CCP specimen (Fig. 3) in plane strain case

Integral path (Fig. 14)	1	2	3	4
Lower bound solutions by $J(10)$	4.404	4.493	4.485	4.483
Upper bound solutions by $I^*(1)$	4.525	4.565	4.555	4.556
Exact K_I	4.506	4.506	4.506	4.506

($E = 1.0$, $\mu = 0.3$, $\sigma_\infty = 1.0$).

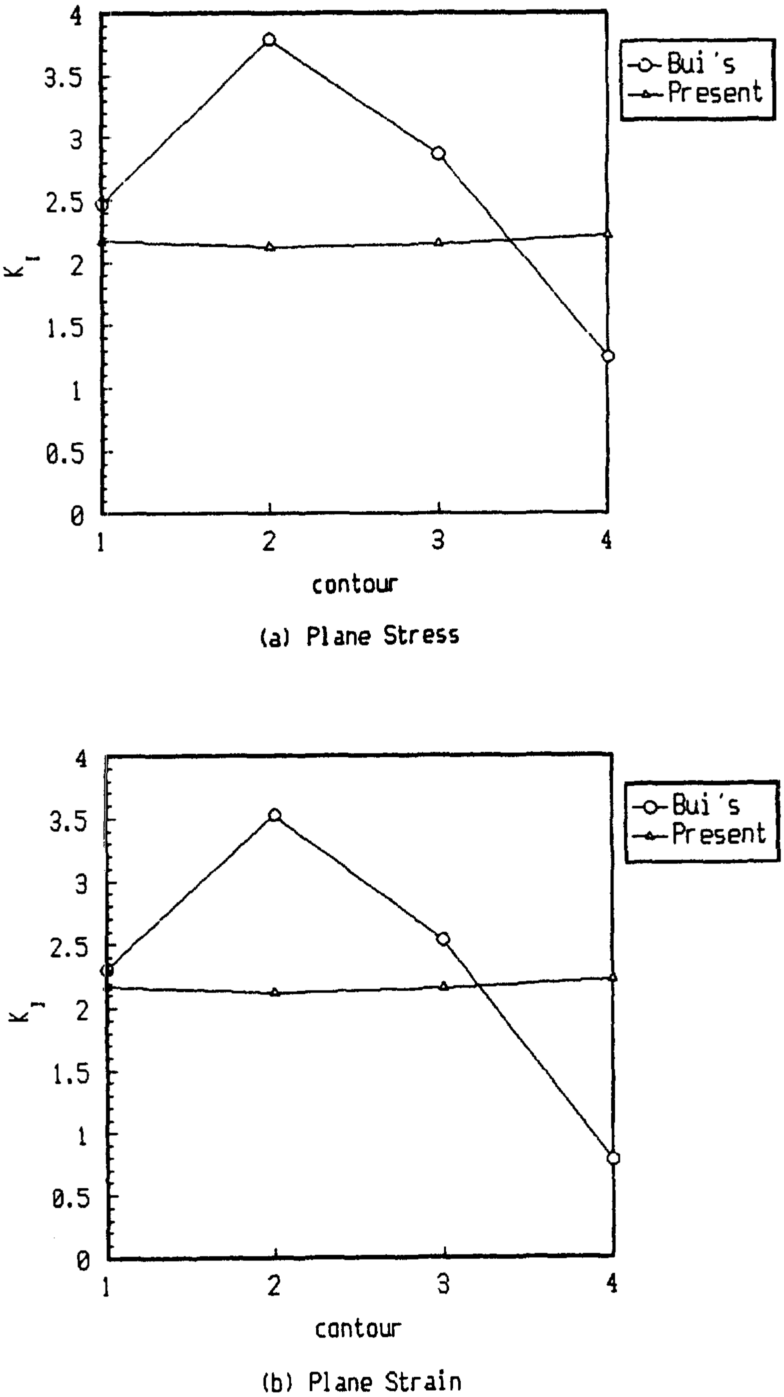

Fig. 13. K_I-solutions of CBB due to eqns (1) and (25) ($K_{I\text{exact}} = 2.185$).

7. CONCLUSIONS

As a dual form of Rice's J-integral, the I^*-integral is developed, which is path independent integral and equivalent to the value of J-integral for a given real status. The mechanical sense of I^* is the complementary energy release rate of nonlinear elastic crack systems. Based on the dual analysis for J and I^*, the upper and lower bound theorems are established, such that the bound estimations for the fracture parameters can easily carry out. It would be very interesting for engineers that I^* is able to provide an approximate upper bound for the exact solution. In addition, it can be expected that the I^*-integral will

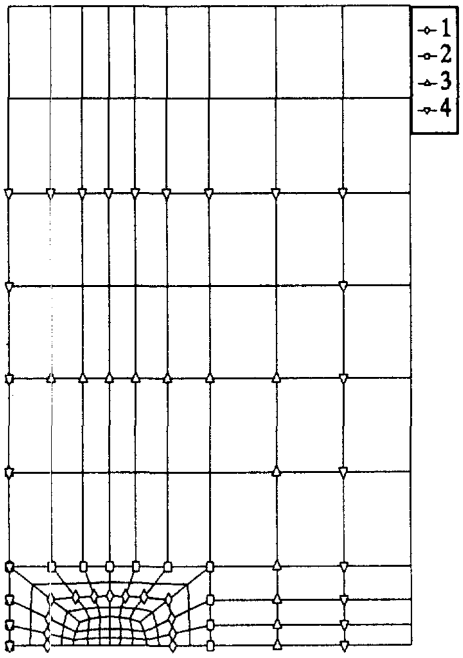

Fig. 14. Meshes and contours for 1/4 CCP specimen used in bound analysis.

provide the hybrid finite elements with a bright prospect in the computational fracture mechanics.

As the end of the paper, let us consider a special case further: the front of crack mn (Fig. 2) is chosen as an integral path for both J and I^*. Since $T_i = \sigma_{ij}n_j = 0$ on mn, we obtain from eqn (10):

$$J = \int_m^n W(\varepsilon_{ij})\,\mathrm{d}x_2 \tag{48}$$

so that J can be explained as an averaged measure of the strain energy on the front of crack. On the other hand, from eqn (1) and by using eqn (31),

$$I^* = \int_m^n B(\sigma_{ij})\,\mathrm{d}x_2 \tag{49}$$

so that I^* can be explained as an averaged measure of the complementary energy on the front of crack. Here the duality of J and I^* is exhibited by eqns (48) and (49) in a perfect version.

Acknowledgements—The financial assistance from Japan Society for the Promotion of Science (JSPS) and the National Natural Science Foundation of China are gratefully acknowledged. The author would also like to thank Dr T. Furukawa for his meaningful suggestions on this project.

REFERENCES

Atluri, S. N. (1982) Path-independent integrals in finite elasticity and inelasticity, with body forces, inertia, and arbitrary crack-face conditions. *Engineering Fracture Mechanics* **16,** 341–364.

Atluri, S. N., Nishioka, T. and Nakagaki, M. (1984) Incremental path-independent integrals in inelastic and dynamic mechanics. *Engineering Fracture Mechanics* **20,** 209–244.

Bui, H. D. (1974) Dual path independent integrals in the boundary-value problems of cracks. *Engineering Fracture Mechanics* **6,** 287–296.

He, M. Y. and Hutchinson, J. W. (1983) Bounds for fully plastic crack problems for infinite bodies. *Elastic–Plastic Fracture: 2nd Symposium—Inelastic Crack Analysis*, Vol. 1, ed. C. F. Shih *et al.* ASTM Special Tech. Pub. 803, pp. 1.177–1.290.

Eshelby, J. D. (1970) The energy momentum tensor in continuum mechanics. *Inelastic Behavior of Solids*, ed. M. F. Kanninen *et al.* McGraw-Hill, New York.

Ewalds, H. L. and Wanhill, R. J. H. (1984) *Fracture Mechanics.* Arnold, London.

Hellen, T. K. and Blackburn, W. S. (1986) Nonlinear fracture mechanics and finite elements. *Numerical Methods for Non-Linear Problems*, Proc. of the Int. Conf. held at Dubrounik, Yugoslavia, 15–18 Sept. 1986. Vol. 3, ed. C. Taylor *et al.*, pp. 3–41.

Kanninen, M. F. and Popelar, C. H. (1985) *Advanced Fracture Mechanics.* Oxford University Press, Oxford.

Knowles, J. K. and Sternberg, E. (1971/72) On a class of conservation laws in linearized and finite elastostatics. *Archive for Rational Mechanics and Analysis* **44,** 187–211.

Nagtegaal, J. C., Parks, D. M. and Rice, J. R. (1974) On numerically accurate finite element solutions in fully plastic range. *Computational Methods of Applied Mechanicsal Engineering* **4,** 153–177.

Olver, P. J. (1984) Conservation laws in elasticity II. Linear homogeneous isotropic elastostatics. *Archive for Rational Mechanics and Analysis* **85**(2), 131–160.

Pian, T. H. H. and Sumihara, K. (1984) Rational approach for assumed stress finite elements. *International Jounral of Numerical Methods of Engineering* **20,** 1685–1695.

Rice, J. R. (1968) A path independent integral and the approximate analysis of strain concentration by notches and cracks. *Journal of Applied Mechanics* **35,** 376–386.

Tsamasphyros, G. J. and Theocaris, P. S. (1982) A new concept of path independent integrals for plane elasticity. *Journal of Elasticity* **12,** 265–280.

Washizu, K. (1975) *Variational Methods in Elasticity and Plasticity*, 2nd edn. Pergamon Press, New York.

Wu, C. C. and Cheung, Y. K. (1995) On optimization approaches of hybrid stress elements. *Finite Elements in Analysis and Design* **21,** 111–128.

Wu, C. C. and Pian, T. H. H. (1997) *Incompatible Numerical Analysis and Hybrid Element Method.* Science Press, Beijing.

Xiao, Q. Z. (1996) Path integrals in nonlinear fracture mechanics and related numerical methodology. Ph.D. thesis, University of Science and Technology of China, Hefei.